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1990 J. Phys. A: Math. Gen. 23 3095

(http://iopscience.iop.org/0305-4470/23/13/043)

J. Phys. A: Math. Gen. 23 (1990) 3095-3097. Printed in the UK

COMMENT

Comment on 'Integrable potentials with logarithmic integrals of motion'

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Received 2 February 1990

Abstract. We analyse and complete the results of an article by Ichtiaroglou and Voyatzis concerning the existence of integrals of motion, logarithmic in the momenta, for two-dimensional, velocity-dependent Hamiltonians. We find a new integrable case, and we establish the generality of some results found by Ichtiaroglou and Voyatzis which were presented as particular cases.

In a recent publication [1] Ichtiaroglou and Voyatzis presented an interesting study concerning the existence of two-dimensional, velocity-dependent Hamiltonians which possess a second invariant logarithmic in the momenta. Their study makes a significant contribution to the domain of transcendental invariants. The quest for non-polynomial invariants in 2D autonomous systems was initiated by Hietarinta in [2] who, in his review article [3], also made a first attempt at a classification of such invariants. It goes without saying that very few results exist to date. In fact very little is known on the integrability of velocity-dependent Hamiltonians [4–7].

In their paper the authors of [1] have analysed the conditions for the existence of a logarithmic invariant. They have presented the general solution in some cases, examples of solutions in some others, while other cases have remained open. Moreover the question of complex potentials was not touched upon. The aim of the present paper is to complete their study.

Ichtiaroglou and Voyatzis consider the integral:

$$I = F_1\dot{x} + F_2\dot{y} + F_3 + \ln(G_1\dot{x} + G_2\dot{y} + G_3) \tag{1}$$

which is invariant under the motion resulting from the equations:

$$\begin{aligned} \ddot{x} &= -U_x + \Omega\dot{y} \\ \ddot{y} &= -U_y - \Omega\dot{x}. \end{aligned} \tag{2}$$

A simple physical interpretation of U and Ω is that of potential and magnetic field respectively.

Elementary considerations for the existence of the invariant lead to the following expressions for the F_i, G_i:

$$\begin{aligned} F_1 &= \delta y + \varepsilon \\ F_2 &= -\delta x + \zeta \\ G_1 &= \alpha y + \beta \\ G_2 &= -\alpha x + \gamma \\ F_3 &= F_3(u) \text{ with } u = \delta(x^2+y^2)/2 + \varepsilon y - \zeta x \end{aligned} \tag{3}$$

and $\Omega = \Omega(u) = -F_{3u}$. The quantity G_3 is computed from:

$$\begin{aligned} G_{3x} &= \Omega G_2 + G_1 W \\ G_{3y} &= -\Omega G_1 + G_2 W \quad \text{with } W = F_1 U_x + F_2 U_y \end{aligned} \tag{4}$$

and one nonlinear compatibility condition remains to be checked:

$$G_3 W + G_1 U_x + G_2 U_y = 0. \tag{5}$$

Rotations, translations and scaling of the variables allow us to reduce the form of the invariant and classify the various cases [3]. Here is the complete list of all non-trivial cases (trivial means that either all F_i or all G_i are identically zero):

I: $\delta = 0$
- IA: $\varepsilon = 1$, $\zeta = 0$
  - IA1: $\alpha = 0$, $\beta\gamma \neq 0$
  - IA2: $\alpha = 1$, $\beta = \gamma = 0$
- IB: $\varepsilon = 1$, $\zeta = \mathrm{i}$
  - IB1: $\alpha = 0$, $\beta = 1$, $\gamma = 0$
  - IB2: $\alpha = 0$, $\beta = 1$, $\gamma = \mathrm{i}$
  - IB3: $\alpha = 0$, $\beta = 1$, $\gamma = -\mathrm{i}$
  - IB4: $\alpha = 1$, $\beta = \gamma = 0$

II: $\delta = 1$, $\varepsilon = \zeta = 0$
- IIA: $\alpha = 0$
  - IIA1: $\beta = 1$, $\gamma = 0$
  - IIA2: $\beta = 1$, $\gamma = \mathrm{i}$
- IIB: $\alpha = 1$
  - IIB1: $\beta = 0$, $\gamma = 0$
  - IIB2: $\beta = 1$, $\gamma = 0$
  - IIB3: $\beta = 1$, $\gamma = \mathrm{i}$.

In [1] the cases involving imaginary ζ or γ have not been considered at all. Among the cases with real coefficients only $\delta = 0$ cases have been analysed. Various subcases were distinguished and solutions were obtained. Two cases need retain our interest: case IA1 ($\delta = 0$, $\varepsilon = 1$, $\zeta = 0$, $\alpha = 0$) in the generic case where both β and γ are non-zero, and case IA2 ($\alpha = 1$, $\beta = \gamma = 0$). In the first case Ichtiaroglou and Voyatzis found (when β is scaled to 1) that $G_3 = (1+\gamma^2)F_3 + f(\gamma y + x)$. They did not proceed to find the general solution for the compatibility equation (5) but limited themselves to the subcases $F_3 = 0$, $f = 0$, $f = \gamma y + x$. In the second case they gave a particular example of realisation with $\Omega = 1/y$.

A first step in our study was to analyse the solutions presented in [1]. We will not go here into any tedious algebra. The net result was that the special cases given by Ichtiaroglou and Voyatzis for IA1 and IA2 were in fact the only possible ones and their analysis had captured the most general solution. We proceeded further to examine the remaining cases. In each one (with the exception of IIB2 and IIB3 where the complexity of the compatibility equation became prohibitive) we were able to complete the analysis (sometimes with the help of a considerable amount of computer algebra). In most cases no new non-trivial integrable Hamiltonian resulted. Still, one new result was found for IB4 ($\delta = 0$, $\varepsilon = 1$, $\zeta = \mathrm{i}$, $\alpha = 0$, $\beta = 1$, $\gamma = -\mathrm{i}$). In this case it is clear that the variables $z = x + \mathrm{i}y$ and z^* are the adequate ones. From (3) we have:

$$\Omega = \Omega(z). \tag{6}$$

Rewriting (4) in these variables we obtain:

$$\begin{aligned} &G_{3z}=0 \\ &G_{3z^*}=-\mathrm{i}\Omega+2\mathrm{i}U_{z^*}. \end{aligned} \tag{7}$$

We get thus:

$$\begin{aligned} &G_3=G_3(z^*) \\ &2U=z^*\Omega(z)-\mathrm{i}G_3(z^*)+h(z). \end{aligned} \tag{8}$$

Next we substitute in (5) which becomes:

$$G_3(z^*)(\mathrm{i}\Omega(z)+G_3'(z^*))+z^*\Omega'(z)+h'(z)=0. \tag{9}$$

By differentiating with respect to z and to z^* we get:

$$\mathrm{i}\Omega'(z)G_3'(z^*)+\Omega''(z)=0. \tag{10}$$

Separation of variables leads us to the solution, which must, of course, be reinjected in the non-differentiated expression (9) before it is considered acceptable. We find thus (A, λ, μ are free constants):

$$\begin{aligned} &G_3=\lambda z^*+\mu \\ &\Omega=2A\lambda\,\mathrm{e}^{-\mathrm{i}z\lambda}+\mathrm{i}\lambda \\ &U=A(\lambda z^*+\mu)\,\mathrm{e}^{-\mathrm{i}z\lambda} \end{aligned} \tag{11}$$

leading to the invariant:

$$I=\mathrm{i}\dot{z}-2\mathrm{i}A\,\mathrm{e}^{-\mathrm{i}z\lambda}-\mathrm{i}\lambda z+\ln(\dot{z}^*+\lambda z^*+\mu). \tag{12}$$

Thus the analysis of the existence of velocity-dependent 2D Hamiltonians with logarithmic invariants involving linear functions of the momenta has been essentially completed. It provided new results, proved that no integrable cases are to be expected for some values of the parameters and established the generality of results found in [1] and presented as mere particular cases.

Acknowledgment

The authors are indebted to Dr G Voyatzis for illuminating discussions.

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